**Question 3.**

Find $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\frac{1}{n}\right)$, where $\ln$ denotes the natural logarithm.

**<u>Solution</u>**

It is known that

$$-1 = \int_0^1 \ln x\, dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right),$$

and

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

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Then,

$$\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\frac{1}{n}\right) \geq \quad \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right) \to -1.$$

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$0 < \frac{1}{n} \leq 1$, for $n_0 = 1$, therefore

$$\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\frac{1}{n}\right) \leq \quad \frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+1\right)$$

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Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+1\right) = \int_0^1 \ln(x+1)dx$$

$$= \int_1^2 \ln x\, dx \approx 0.38629$$

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We obtain the result as $n \to \infty$, $\frac{1}{n} \to 0$, therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}+\frac{1}{n}\right) = -1$$